Spanning trees with small degrees and few leaves☆

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Abstract

We give an Ore-type condition sufficient for a graph G to have a spanning tree with small degrees and with few leaves.

Keywords:
Spanning tree. Bounded degree. Few leaves.

1. Introduction

From a classical result by Ore [3] it is well-known that if a simple graph G with $n \geq 2$ vertices is such that $d(u) + d(v) \geq n - 1$ for each pair u, v of non-adjacent vertices of G, then G contains a hamiltonian path.

A *leaf* of a tree T is a vertex of T with degree one. A natural generalisation of hamiltonian paths are spanning trees with a small number of leaves. In this direction, Ore's result was generalised by Broersma and Tuinstra [1] to the following theorem.

Theorem 1.1. [1] *Let $s \geq 2$ and $n \geq 2$ be integers. If G is a connected simple graph with n vertices such that $d(u) + d(v) \geq n - s + 1$, for each pair u, v of non-adjacent vertices, then G contains a spanning tree with at most s leaves.*

Further related results have been obtained by Egawa *et al* [2] and by Tsugaki and Yamashita [5]. See also [4] for a survey on spanning trees with specific properties.

In this note we consider spanning trees with small degrees as well as with a small number of leaves. Our result is the following.

Theorem 1.2. *Let n, k and $d_1, d_2, \ldots, d_n$ be integers with $1 \leq k \leq n-1$ and $2 \leq d_1 \leq d_2 \leq \cdots \leq d_n \leq n-1$. If G is a k-connected simple graph with vertex set $V(G) = \{w_1, w_2, \ldots, w_n\}$ such that $d(u) + d(v) \geq n - 1 - \sum_{j=1}^{k} (d_i - 2)$ for*

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any non-adjacent vertices u and v of G, then G has a spanning tree T with at most $2+\sum_{j=1}^{k}(d_j-2)$ leaves and such that $d_T(w_j)\le d_j$ for $j=1,2,\ldots,n$.

2. Proof of Theorem 2

Let T be a largest subtree of G with at most $2+\sum_{j=1}^{k}(d_j-2)$ leaves and such that if $w_j\in V(T)$, then $d_T(w_j)\le d_j$. Since G is k-connected and $n\ge 2$, it contains a path with at least $k+1$ vertices. Therefore, we may assume that tree T has at least $k+1$ vertices.

If T is not a spanning tree, there is a vertex w of G not in T. By Menger's theorem, there are k internally disjoint paths $\pi_1,\pi_2,\ldots,\pi_k$ in G joining w to k different vertices $r_1,r_2,\ldots,r_k$ of T.

Let n_1 denote the number of leaves of T. We claim $n_1=2+\sum_{j=1}^{k}(d_j-2)$, otherwise there is a vertex r_i such that $d_T(r_i)<d_{j_i}$ where $w_{j_i}=r_i$. Then $T'=T\cup\pi_i$ is a subtree of G with more vertices than T such that $d_{T'}(w_j)\le d_j$ for each $w_j\in V(T')$ and with at most $n_1+1\le 2+\sum_{j=1}^{k}(d_j-2)$ leaves, which contradicts our assumption on the maximality of T.

Because of Ore's theorem, we can assume $d_i\ge 3$ for some $i=1,2,\ldots,k$. Since T has $n_1=2+\sum_{j=1}^{k}(d_j-2)\ge 3$ leaves, as shown above, there is a vertex w_j of T such that $d_T(w_j)\ge 3$. Suppose there are vertices x and y of degree one in T such that $xy\in E(G)$. Since T is not a path, there is an edge zz' in the unique xy path contained in T with $d_T(z)\ge 3$. Let $T'=(T-zz')+xy$ and notice that T' is a subtree of G with $V(T')=V(T)$, with less than $2+\sum_{j=1}^{k}(d_j-2)$ leaves and such that $d_{T'}(w_j)\le d_j$ for each $w_j\in V(T')$. As above, this is a contradiction and therefore no leaves of T are adjacent in G.

Notice that $d_T(r_1)\ge 2$, otherwise $T'=T\cup\pi_1$ would be a tree larger than T, with the same number of leaves and with $d_{T'}(w_j)\le d_j$ for each vertex w_j of T'. Let u and v be any two leaves of T with the property that the vertex r_1 lies in the unique uv path T_{uv}, contained in T. Orient the edges of T in such a way that the corresponding directed tree $\overrightarrow{T}$ is outdirected with root u (see Fig. 1.)

For each vertex $z\ne u$ in T let z^- be the unique vertex of T such that z^-z is an arc of $\overrightarrow{T}$. Let

$$A=\{y\in V(T):yv\in E(G)\}\text{ and }B=\left\{x^-\in V(T):ux\in E(G)\right\}.$$

Because of the way the tree T was chosen, all vertices of G adjacent to u or to v lie in T and therefore $|A|=d(v)$. Let x_1 and x_2 be vertices of T adjacent to u in G , if $x_1^-=x_2^-=z$ for some vertex z of T, let $T'=(T+ux_1)-zx_1$. Since zx_1 and zx_2 are edges of T, $d_{T'}(z)\ge 2$ and T' is a subtree of G with $V(T')=V(T)$,

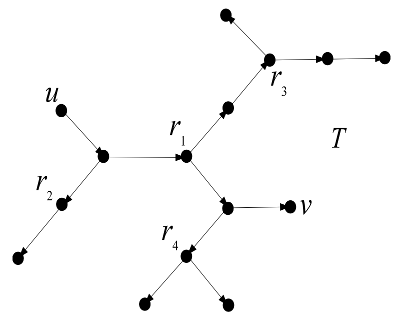

Figure 1: $n = 15$, $k = 4$, $d_1, d_2, \ldots, d_{15} = 3$.

with less than $2 + \sum_{j=1}^{k} \left(d_{i_j} - 2\right)$ leaves and such that $d_{T'}(w_j) \leq d_j$ for each $w_j \in V(T')$. Again, this is a contradiction, therefore $|B| = d(u)$.

Since no vertex in $A \cup (B \setminus \{u\})$ is a leave of T,

$$|A \cup B| \leq |V(T)| - n_1 + 1 \leq (n-1) - n_1 + 1 = n - 2 - \sum_{j=1}^{k} (d_j - 2).$$

Also

$$|A \cup B| = |A| + |B| - |A \cap B| = d(u) + d(v) - |A \cap B| \geq \\ n - 1 - \sum_{j=1}^{k} (d_j - 2) - |A \cap B|.$$

Therefore $|A \cap B| \geq 1$; let $z^- \in A \cap B$. We consider two cases:

Case 1. Edge $z^- z$ lies on the path T_{uv}.

If $z = r_1$(see Fig. 2), let

$$T' = \left(\left(T + z^- v\right) - z^- z\right) \cup \pi_1 \text{ and}$$

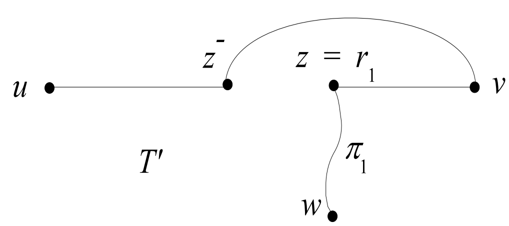

Figure 2: $T' = \left(\left(T + z^- v\right) - z^- z\right) \cup \pi_1$

and if $r_1 \neq z$ (see Fig. 3), let

$$T' = \left(\left(\left(\left(T + uz\right) + z^- v\right) - r_1^- r_1\right) - z^- z\right) \cup \pi_1.$$

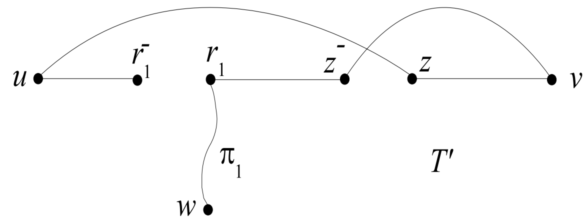

Figure 3: $T' = \left(\left(((T+uz)+z^-v) - r_1^- r_1\right) - z^- z\right) \cup \pi_1$

Both situations lead to a contradiction since T' is a subtree of G larger than T, with at most $2 + \sum\limits_{j=1}^{k} (d_j - 2)$ leaves and such that $d_T(w_j) \leq d_j$ for each $w_j \in V(T')$.

Case 2. Edge $z^- z$ does not lie on the path T_{uv}.

If z^-lies in T_{uv}, let $T'' = (T + uz) - z^- z$ (see Fig. 4).

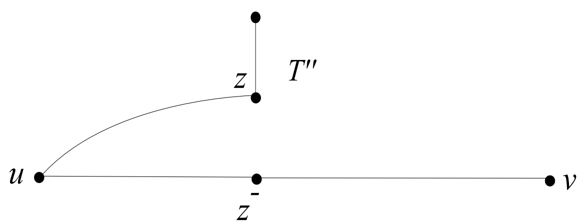

Figure 4: $T'' = (T + uz) - z^- z$

And if z^- does not lie in T_{uv}, let x be a vertex in T_{uv} not in T_{uz^-} such that x^- is a vertex in T_{uz^-} (see Fig. 5). Let

$$T'' = \left(\left(((T+uz)+z^-v\right) - x^-x\right) - z^-z\right)$$

In this case T'' is a subtree of G with $V(T'') = V(T)$, with at most $n_1 - 1 = 1 + \sum\limits_{j=1}^{k} (d_j - 2)$ leaves and such that $d_{T''}(w_j) \leq d_j$ for each $w_j \in V(T'')$. As seen above, this is not possible.

Cases 1 and 2 cover all possibilities, therefore T is a spanning tree of G.

Let $k \geqslant 1$ and $d_1, d_2, \ldots, d_n$ be integers with $3 \leq d_1 \leq d_2 \leq \cdots \leq d_n$ and $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_{2-k+d_1+\cdots+d_k}\}$ be sets of vertices. The complete bipartite graph G with bipartition (X, Y) is k-connected, has $n = 2 + \sum\limits_{j=1}^{k} d_i$ vertices and is such that $d(u) + d(v) \geq 2k = n - 2 - \sum\limits_{j=1}^{k} (d_i - 2)$ for any vertices u and v of G. Nevertheless, if T is a spanning tree of G, then

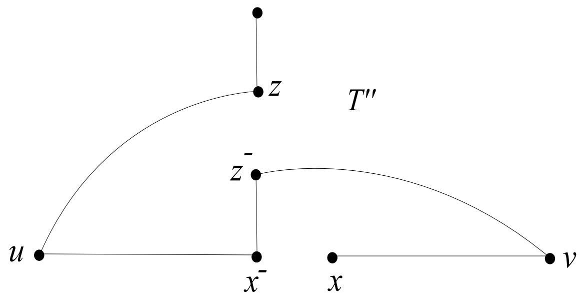

Figure 5: $T'' = \left(\left(\left((T + uz) + z^- v\right) - x^- x\right) - z^- z\right)$

$d_T(x_j) > d_j$ for some $j = 1, 2, \ldots, k$. This shows that the condition in Theorem 2 is tight.

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